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1984 J. Phys. A: Math. Gen. 17 L609

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LETTER TO THE EDITOR

**Diluted continuous spin models near the percolation threshold**

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Received 15 May 1984

**Abstract.** The crossover exponents describing the behaviour of continuous spin models at dilutions near the percolation threshold are calculated within an  $\epsilon$  expansion, where  $\epsilon = 6 - d$ , where  $d$  is the spatial dimensionality. Our result confirms a recent calculation for the random resistor network and shows that earlier calculations of the crossover exponents are incorrect.

When models exhibiting phase transitions are randomly diluted, the transition temperature  $T_c(p)$  decreases monotonically with the concentration  $p$  of occupied sites (or bonds). At the percolation threshold  $p = p_c$ , an infinite connected cluster of occupied sites ceases to exist (for a review of percolation, see Essam (1980)), and  $T_c(p) = 0$ . The point  $P: p = p_c, T = 0$  is thus a multicritical point where long-range thermodynamic and geometric order develop simultaneously (Stephen and Grest 1977, referred to as SG). The critical properties in the vicinity of  $P$  for models (such as the Ising model) possessing an energy gap  $J$  are well understood: there is a single crossover exponent  $\varphi = 1$  (SG, Wallace and Young 1978) associated with  $w = e^{-J/T}$  so that, for example, the order parameter susceptibility  $\chi$  satisfies the scaling relation  $\chi = |p - p_c|^{-\gamma_p} f(w/|p - p_c|)$  where  $\gamma_p$  is the percolation exponent measuring the mean square cluster size. In this paper, we will consider the critical properties in the vicinity of  $P$  of a class of models (such as the  $xy$  model) whose energy spectrum does not possess an energy gap. We obtain results consistent with those recently found (Harris, Kim and Lubensky, to be published, referred to as HKL) for the randomly diluted  $q$ -state Potts model in the limit  $q \rightarrow 0$ .

We begin with models on a  $d$ -dimensional hypercubic lattice with a scalar variable  $\vartheta(x)$  at each site  $x$  governed by an interaction Hamiltonian

$$H = - \sum_{\langle x, x' \rangle} U[\vartheta(x) - \vartheta(x')], \tag{1}$$

where  $\langle x, x' \rangle$  signifies the bond connecting  $x$  to  $x'$  and  $U(\vartheta)$  is a model-dependent function. If  $U(\vartheta) = K \cos \vartheta$  with  $0 \leq \vartheta \leq 2\pi$ , then (1) is the  $xy$  model; if  $U(V) = -\frac{1}{2}KV^2$  where  $-\infty < V < \infty$ , then (1) describes a resistor network (Stephen 1978) with conductances  $K$  on each bond and  $V(x)$  is associated with the voltage at point  $x$ . In the former case,  $\langle \cos \vartheta(x) \rangle = \text{Re}(e^{i\vartheta(x)})$  is the  $xy$  order parameter, and in the latter case,  $R(x, x') = \langle (V(x) - V(x'))^2 \rangle$  is the resistance between the sites  $x$  and  $x'$  where  $\langle A \rangle = \int d\vartheta A e^{-H} / \int d\vartheta e^{-H}$ .

Properties of quenched random versions of (1) in which bonds are present with probability  $p$  and absent with probability  $1 - p$  are described in the usual way by an

effective Hamiltonian  $H_{\text{eff}}$  obtained by  $n$  replications of  $H$  in the limit  $n \rightarrow 0$ :

$$\exp(-H_{\text{eff}}) = \left[ \exp\left(-\sum_{\alpha=1}^n H_{\alpha}\right) \right]_p, \quad (2a)$$

$$H_{\text{eff}} = \sum_{\langle x, x' \rangle} \ln \left[ 1 - p + p \exp\left(\sum_{\alpha} U[\vartheta_{\alpha}(x) - \vartheta_{\alpha}(x')]\right) \right], \quad (2b)$$

where  $[ ]_p$  signifies an average with respect to the random distribution of occupied bonds and  $\alpha$  is a replica index. To study the point  $P$ , we follow Stephen's (1978) treatment of the random resistor network. We first introduce a discrete version of (1) in which  $\vartheta$  takes on a discrete set of values separated by an interval  $\Delta\vartheta$ . This step allows us to define precisely variables related to percolation. We will regain the original model by allowing  $\Delta\vartheta \rightarrow 0$  at the appropriate point in calculations. We now define order parameters

$$\psi_{\mathbf{k}} = e^{i\mathbf{k} \cdot \boldsymbol{\vartheta}}, \quad \mathbf{k} \neq 0, \quad (3)$$

where  $\mathbf{k} = (k_1, \dots, k_n)$  and  $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_n)$ , where the  $k$ th component refers to the  $k$ th replica. In terms of these variables, we have

$$H_{\text{eff}} = \sum_{\langle x, x' \rangle} \sum_{\mathbf{k}} B_{\mathbf{k}} \psi_{\mathbf{k}}(x) \psi_{\mathbf{k}}(x')^*, \quad (4)$$

where

$$B_{\mathbf{k}} = \sum_{l=1}^{\infty} \frac{(-1)^l}{l} v^l \prod_{\alpha=1}^n F_l(k_{\alpha}) \quad (5)$$

with  $v = p/(1-p)$  and

$$F_l(k) = \sum \frac{\Delta\vartheta}{2\pi} e^{-ik\vartheta} e^{lU(\vartheta)} \rightarrow \int \frac{d\vartheta}{2\pi} e^{-ik\vartheta} e^{lU(\vartheta)}. \quad (6)$$

For the resistor network,  $U$  is quadratic in  $\vartheta$ , and  $\vartheta$  extends between  $-\infty$  and  $\infty$ , so that

$$\prod_{\alpha} F_l(k_{\alpha}) = \exp[-k^2/(2lK)] \quad (7)$$

is only a function of the rotationally invariant quantity  $k^2 = \sum_{\alpha} k_{\alpha}^2$ . For the  $xy$  model, the leading large- $K$  part of  $\prod F_l(k_{\alpha})$  is given by (7), but because  $K(\vartheta)$  contains terms of order  $\vartheta^4$  and higher, there will be corrections to (7) involving 'cubic' invariants of  $\mathbf{k}$  such as  $\sum_{\alpha} k_{\alpha}^4$ .

Since  $H_{\text{eff}}$  in (4) is quadratic in each  $\psi_{\mathbf{k}}$ , we can apply the Stratanovich transformation to obtain a field theory in terms of fields  $\varphi_{\mathbf{k}}$  conjugate to  $\psi_{\mathbf{k}}$ :

$$Z = \text{Tr} e^{-H_{\text{eff}}} = \int D\varphi e^{-L}, \quad (8)$$

where  $D\varphi$  indicates integration over all  $\varphi$ 's and

$$L = \frac{1}{2} \int d^d x \sum'_{\mathbf{k}} [r_{\mathbf{k}} \varphi_{\mathbf{k}}(x) \varphi_{-\mathbf{k}}(x) + \nabla \varphi_{\mathbf{k}}(x) \cdot \nabla \varphi_{-\mathbf{k}}(x)] \Delta \mathbf{k} \\ + (1/3!) u \int d^d x \sum'_{\mathbf{k}_1, \mathbf{k}_2} \varphi_{\mathbf{k}_1}(x) \varphi_{\mathbf{k}_2}(x) \varphi_{-\mathbf{k}_1 - \mathbf{k}_2}(x) \Delta \mathbf{k}_1 \Delta \mathbf{k}_2, \quad (9)$$

where the primes indicate omission of terms for which any  $k$  vanishes and  $r_k = 1 - (zB_k)^{-1}$ , where  $z$  is the number of nearest neighbours. In (9) we have omitted terms of higher order than  $\phi^3$ , since these are irrelevant for  $d$  near  $d_c = 6$ . When  $K = \infty$  (i.e. at  $P$ ),  $F_i(k_\alpha) = 1$ , so that all  $r_k$  are equal to  $r_0 \sim p - p_c$ . When  $K^{-1} > 0$ , we can expand  $r_k$  in powers of  $k$ :

$$r_k = r_0 + \sum_m w_m k^{2m} + v_4 \sum_\alpha k_\alpha^4 + \dots, \tag{10}$$

where the terms left out are higher-order terms of cubic rather than spherical symmetry. Here  $w_m$  may be viewed as being of order  $K^{-m}$ .

We are now in a position to develop momentum shell recursion relations (Rudnick and Nelson 1976) in  $6 - \epsilon$  dimensions. When all the  $w_m$  and  $v$ 's are zero, one regains the familiar percolation results (Harris *et al* 1975) with the third-order vertex reaching a fixed point value of  $u^2 = \epsilon/7$ . This describes the behaviour at  $K^{-1} = 0$  ( $T = 0$ ). When  $K^{-1} > 0$ , we have

$$d r_k / d l = (2 - \eta_p) r_k - u^2 \Sigma_k, \tag{11}$$

where  $\eta_p = -\epsilon/21$  is the percolation exponent, and

$$\Sigma_k = -2G(k)G(0) + \sum_p \Delta p G(p)G(p+k), \tag{12}$$

where  $G(p) = (r_p + q^2)^{-1}$  evaluated at momentum  $q = 1$  is the contribution from the one-loop diagram shown in figure 1. It is important to remember that  $\psi_k$  is an order parameter only for  $k \neq 0$ . Thus there can be no lines in any diagram with  $k = 0$ . In (12) the terms with  $p = 0$  or  $p + k = 0$  which are included in the sum over  $p$  have been explicitly removed by the first term. Here  $G(0)$  is the propagator  $G(k)$  evaluated at  $k = 0$ . It does not, however, represent the propagator of a physical order parameter without further interpretation. Equation (12) can conveniently be manipulated into the form

$$\Sigma_k = -2G(k)G(0) + G^2(0) + \delta \Sigma_k, \tag{13}$$

where

$$\delta \Sigma_k = -\frac{1}{2} \sum_p [G(p+k) - G(p)]^2 \Delta p. \tag{14}$$

Since  $G(p+k) - G(p)$  is of order  $K^{-1}$  for small  $K^{-1}$ , one is tempted to conclude that  $\delta \Sigma_k \sim K^{-2}$  may be ignored in the linearised recursion relations which give the crossover exponent associated with  $K^{-1}$ . Indeed, when there is a gap in the energy spectrum, this is the correct procedure. In that case, there is a single crossover exponent  $\varphi = 1$  obtained (by SG) from the first two terms of (13). When there is no gap in the energy

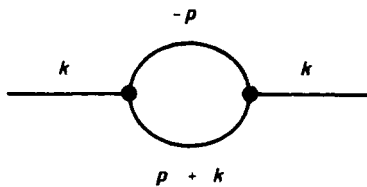


Figure 1. One-loop contribution to  $\Sigma_k$  of equation (11).

spectrum, there are of order  $K$  terms which contribute to the sum in (14) so that  $\delta\Sigma_k \sim K^{-1}$ . In this case, recursion relations for the  $w_m$ 's and  $v$ 's appearing in (10) must include the contribution obtained by expanding (14) in powers of  $k$ . Since  $\delta\Sigma_k$  only gives contributions to  $w_k$  from  $w_m$ 's with  $m > k$ , the crossover exponents are determined by the coupling of  $w_k$  to itself. Then we write

$$dw_k/dl = [2 - \eta_p - u^2(2 + c_k)]w_k, \quad (15)$$

where, after some tedious algebra, we obtain

$$c_k = \frac{1}{k!} \frac{(-)^k}{(k-1)!} \int_0^\infty y^{k-1} \left[ \left( \frac{d}{dy} \right)^k \frac{1}{1+y^k} \right]^2 dy, \quad (16)$$

which gives  $c_1 = -\frac{1}{3}$  and  $c_2 = \frac{1}{5}$ . The crossover exponent associated with  $w_k$  satisfies

$$\varphi_k = 1 - c_k \varepsilon / 14. \quad (17)$$

These results duplicate those obtained by HKL using the zero-state Potts model formulation of the random resistor network. The potentials  $w_k$  in (15) refer only to the rotationally invariant parts of  $r_k$ . There are, in addition, all the potentials that do not preserve rotational invariance. For example, we find a recursion relation for  $v_4$  of the form of (15) leading to an exponent  $\varphi_{v_4} = 1 - \alpha\varepsilon/14$ , where  $\alpha = (5 - 6\Gamma(3/4)^2/\Gamma(1/4)^2)/30$ , where  $\Gamma(x)$  is the gamma function.

We have just shown that there are an infinite number of crossover exponents associated with the percolation multicritical point of gapless models that are of order  $1 + O(\varepsilon)$ . The physical interpretation of these exponents is as follows. We consider first the random resistor network and define

$$\chi_k(\mathbf{x}, \mathbf{x}') \equiv \langle \psi_k(\mathbf{x}) \psi_{-k}(\mathbf{x}') \rangle \quad (18a)$$

$$= [\exp(-\frac{1}{2}k^2 R(\mathbf{x}, \mathbf{x}'))] \quad (18b)$$

$$= |\mathbf{x} - \mathbf{x}'|^{-(d-2+\eta_p)} Y(\{w_m k^m | p - p_c |^{-\varphi_m}, |\mathbf{x} - \mathbf{x}'|/\xi\}), \quad (18c)$$

where  $\xi \sim |p - p_c|^{-\nu}$  is the correlation length, and  $Y$  is some scaling function. Equation (18b) follows because the Hamiltonian for the resistor network is Gaussian and  $\langle (V(\mathbf{x}) - V(\mathbf{x}'))^2 \rangle = R(\mathbf{x}, \mathbf{x}')$  in each replica. Thus  $\chi_k$  is a function of  $k^2 = \Sigma_\alpha k_\alpha^2$  only and consequently depends only on the potentials  $w_m$ . Expanding (18b) in powers of  $k^2$ , we see that the exponents  $\varphi_m$  determine the behaviour of the cumulants of the resistance  $R(\mathbf{x}, \mathbf{x}')$  at  $p_c$ :

$$[R^m(\mathbf{x}, \mathbf{x}')]_c \sim |\mathbf{x} - \mathbf{x}'|^{-(d-2+\eta_p) + (\varphi_m/\nu_p)}. \quad (19)$$

The first cumulant is the average resistance between  $\mathbf{x}$  and  $\mathbf{x}'$ , the second cumulant is  $[R^2(\mathbf{x}, \mathbf{x}')]_p - [R(\mathbf{x}, \mathbf{x}')]_p^2$  and so forth. That the crossover exponents should be unity in mean field theory follows by consideration of a more general model in which occupied bonds are randomly given either of two values of conductance. In this case, when the effect of parallel paths is ignored, one expects  $R(\mathbf{x}, \mathbf{x}')$  to be governed by a Gaussian probability distribution, apart from differences in amplitudes. This reasoning shows that within mean field theory the  $\varphi$ 's are unity. In the  $xy$  and other models,  $U(\vartheta)$  is not quadratic, and it is not possible to express  $\langle \vartheta^{2p} \rangle$  in terms of  $\langle \vartheta^2 \rangle$  only. Thus, there are different types of higher-order cumulants. For example, at order four, we need to

distinguish between  $[\langle(\vartheta(x) - \vartheta(x'))^2\rangle_p^2] - [\langle(\vartheta(x) - \vartheta(x'))^2\rangle_p^2]$  associated with  $w_2$  and  $\varphi_2$  and  $[\langle(\vartheta(x) - \vartheta(x'))^4\rangle_p] - 3[\langle(\vartheta(x) - \vartheta(x'))^2\rangle_p^2]$  associated with  $v_4$  and  $\varphi_{v_4}$ .

The above results for  $\varphi_1$  can also be obtained by studying the analogous crossover behaviour of the  $m$ -component Heisenberg (i.e.  $O(m)$ ) model with  $H = -K \sum_{x,x'} \mathbf{S}(x) \cdot \mathbf{S}(x')$ . We expect  $\varphi_1$  to be independent of  $m$  in view of the one-to-one correspondence between Kirchhoff's equations for the resistor network in the presence of an imposed voltage and those for the equilibrium of the transverse spin components of a ferromagnet in the presence of an imposed magnetisation gradient. For the  $O(m)$  model, the order parameter in (3) is generalised by replacing  $\exp(ik\vartheta)$  in each replica by the set of real-valued orthogonal  $k$ th-order polynomials  $Q_k^\mu(S_1(x), S_2(x), \dots, S_m(x))$  in the variables  $S_i(x)$  with  $\sum_i S_i^2(x) = 1$ . For  $m = 3$  the  $Q_k^\mu$  are essentially the usual spherical harmonics. For general  $m$  there are  $n_k$   $k$ th-order polynomials (with angular momentum  $k$ ), where  $n_k = (2k + m - 2)(k + m - 3)!/[k!(m - 2)!]$ . In this generalisation each index  $k_i$  in (4)–(6) in the above theory is replaced by the pair of indices  $(k_i, \mu_i)$ . For  $K \rightarrow \infty$  we find (for  $m \neq 1$ )

$$F_i(k, \mu) = A\{1 - [k^2 + (m - 2)k]/(m - 1)K + O(K^{-2})\}, \quad (20)$$

where  $A$  is an unimportant constant. To order  $K^{-1}$  we therefore write

$$r_k = r_0 + w_1 \sum_{k_i} [k_i^2 + (m - 2)k_i]. \quad (21)$$

To obtain the recursion relations for  $w_1$  it suffices to study the diagram of figure 1 for small incident angular momentum,  $k$ , but allowing arbitrary angular momentum in the internal legs. In this way we obviate a complete analysis of the Clebsch–Gordan coefficients which appear in the cubic term in the analogue of (9). Otherwise the calculation is very similar to that for  $m = 2$  described above and the result is again  $c_1 = -\frac{1}{3}$ , independent of  $m$ . (In particular we checked that the expression for  $c_1$  can be continued to  $m = 0$ .) For  $m = 1$  the expansion in (21) is clearly invalid, and the usual result  $\varphi_1 = 1$ , i.e.  $c_1 = 0$ , should be adopted.

We may summarise our results as follows. We have analysed the multicritical point for diluted continuous spin models at zero temperature and the percolation threshold within an  $\varepsilon$  expansion near six dimensions. We find a hierarchy of crossover exponents describing various cumulant correlation functions. Our results agree with those found recently (HKL) for the Potts model formulation of the randomly diluted resistor network. Our result that  $\varphi_1 = 1 + \varepsilon/42$  disagrees with previous work (Dasgupta *et al* 1978, Stephen 1978) which found  $\varphi_1 = 1$  to order  $\varepsilon^2$ . In fact the work of Wallace and Young (1978) seemed to prove that  $\varphi_1 = 1$  held to all orders in  $\varepsilon$ . Apparently these works all assume, at least tacitly, that the anomalous term in (14) is of order  $K^{-2}$ , in which case it would be negligible. Our result for  $\varphi_1$  has important and reasonable consequences. Since the exponent  $t$  describing the threshold behaviour of the conductivity,  $\Sigma$ , i.e.  $\Sigma \sim (p - p_c)^t$ , is given by  $t = (d - 2)\nu_p + \varphi_1$  (Harris and Fisch 1977, Dasgupta *et al* 1978, Stephen 1978), numerical evidence (Li and Strieder 1982) that  $t \approx 1.28$  for  $d = 2$  indicates that  $\varphi_1 \neq 1$  for  $d = 2$ . Our result that  $\varphi_1$  varies weakly near  $d = 6$  presumably indicates that it depends weakly and smoothly on  $d$  for  $2 \leq d \leq 6$ . As we discuss elsewhere (HKL), our result for  $\varphi_1$  indicates that a recent conjecture of Alexander and Orbach (1982) fails at first order in  $\varepsilon$ .

This work was supported in part by the National Science Foundation under grant #DMR-19216 and by the Office of Naval Research under grant #ONR0158.

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